Last time:

Newton polygons

Aim: $K$ complete, non-arch. valued field

$g, f \in K[x]$

$\Rightarrow$ If $f$ & $g$ are suff. "close" then the roots of $f$ & $g$ generate the same field ext. of $K$

E.g.: $L/\mathbb{Q}_p$ finite

$$\Rightarrow \exists\, g(x) \in \underline{\underline{\mathbb{Q}}}[x], \text{ s.t. } L \cong \mathbb{Q}_p[x]/(g(x))$$

(not $\mathbb{Q}_p$)

a completion of a number field at the valuation ass. with some maximal ideal (e.g. of $\mathbb{Q}[x]/(g(x))$ :)

Lemma (Krasner): $\alpha, \beta \in K^{sep}$, s.t.

$$|\beta - \alpha| < |\beta - \beta'| \quad \forall \text{ Galois conj. } \beta' \text{ of } \beta, \beta' \neq \beta$$

$$\Rightarrow \quad \beta \in K(\alpha)$$

Proof: $K(\alpha) = (K^{sep})^H$

with $H = \{ \sigma \in Gal(K^{sep}/K) \mid \sigma(\alpha) = \alpha \}$

$\Rightarrow$ STP: $\sigma(\beta) = \beta \quad \forall \sigma \in H$

Now:

$$|\sigma(\beta) - \beta| = |\sigma(\beta) - \alpha + \alpha - \beta|$$

$$\leq \max\{ |\sigma(\beta - \alpha)|, |\alpha - \beta| \}$$

$$\underset{\sigma \in H}{=} |\beta - \alpha| \underset{\text{Ass.}}{<} |\sigma(\beta) - \beta| \text{ if } \sigma(\beta) \neq \beta$$

$\Rightarrow \sigma(\beta) = \beta \quad \forall \sigma \in H$ $\square$

Recall: $\|f\| := \max_{i=0,\dots,n} |a_i|$ $\qquad f(x) = \sum_{i=0}^{n} a_i x^i$

"Gauß norm"

Thm: Let $f(x) \in K[x]$ be irreducible, separable, monic of deg $n$

Let
$$d_0 := \min_{\substack{\alpha \neq \alpha' \\ \alpha, \alpha' \text{ roots of } f \text{ in } K^{sep}}} \{ |\alpha - \alpha'| \}$$

Let $0 < \varepsilon < d_0$

Then there exists $\delta > 0$, s.t.

if $g \in K[x]$ is monic of degree $n$

with $\|f - g\| < \delta$, then there exists

an ordering $\alpha_1, \dots, \alpha_n$ of the roots of $f(x)$

$\beta_1, \dots, \beta_n$ $\qquad g(x)$

& $|\alpha_i - \beta_i| < \varepsilon$, $K(\alpha_i) = K(\beta_i)$

In part, $g$ is irreducible (as $K(\beta_i)$ has deg $n$).

Appl.: $L/\mathbb{Q}_p$ finite of deg $n$

$\Rightarrow \exists f \in \mathbb{Q}_p[x]$, s.t.

$L \cong \mathbb{Q}_p[x]/(f(x))$ ($L/\mathbb{Q}_p$ separable)

For $g(x) \in \mathbb{Q}[x]$ suff. close, monic of degree $n$, $L \cong \mathbb{Q}_p[x]/(g(x))$

In part, $L \cong K_{\mathfrak{p}}$ for some number field $K$ and $\mathfrak{p} \subseteq \mathcal{O}_K$ max'l

Proof of thm:

If $h(x) = x^n + \sum_{i=0}^{n-1} c_i x^i \in K[x]$ monic

& $\gamma \in \overline{K}$ root of $h$,

then $|\gamma|^n = |\sum_{i=0}^{n-1} c_j \gamma^j|$

$\Rightarrow \exists j,$ s.t. $|\gamma|^n \le |c_j \cdot \gamma^j|$

$$\Rightarrow |\gamma| \le \max_{0 \le j \le n-1} |c_j|^{\frac{1}{n-j}}$$

$$\le \max_{0 \le j \le n-1} \|h\|^{\frac{1}{(n-j)}}$$

i.e. $|\gamma|$ is bdd in terms of $\|h\|$ (and $n$)

Let $\tilde{\delta} > 0$, s.t. $0 < \tilde{\delta} \le \|f\|$

Let $g \in K[x]$ monic of degree $n$,

(note if $\|f-g\|$ is suff. small, then

$g$ is separable)

s.t. $\|f - g\| < \tilde{\delta}$

$$\Rightarrow \|g\| \le \max\{\|f\|, \underbrace{\|f-g\|}_{<\tilde{\delta} \le \|f\|}\} \le \|f\|$$

$\Rightarrow \exists$ constant $C_0 > 0$ (depending only $\|f\|$), s.t. $|\beta| < C_0$ for all $\tilde{\delta} \leq \|f\|$, $g$ as above

$\beta \in K^{sep}$ root of $g$

$$\Rightarrow \prod_{\alpha} |\beta - \alpha| = |f(\beta)| = |f(\beta) - g(\beta)|$$

$$\leq C_1 \cdot \|f - g\| \leq C_1 \cdot \tilde{\delta}$$

for some constant $C_1 > 0$ (depending only on $f$)

In part, $\min\limits_{\alpha \text{ root of } f} \{|\beta - \alpha|\} \to 0$ if $\tilde{\delta} \to 0$

In part, $\min\limits_{\alpha \text{ root of } f} \{|\beta - \alpha|\} < \varepsilon$ for some $\delta := \tilde{\delta} > 0$ suff. small

Let $g \in K[x]$ monic, irred. $\|f - g\| < \delta$

$\beta$ root of $g$

Claim: $\exists$ unique root $\alpha(\beta)$ of $f(X)$,

s.t. $|\beta - \alpha|$ minimal

Prf of claim: If $|\beta - \alpha| = |\beta - \alpha'|$ minimal ($\alpha, \alpha'$ roots of $f$), then both are less than $\varepsilon < d_0 = \max\limits_{\substack{\alpha,\alpha' \\ \text{roots of} \\ f,\, \alpha \neq \alpha'}} |\alpha - \alpha'|$

$\Rightarrow |\alpha - \alpha'| \leq \max\{|\beta - \alpha|, |\beta - \alpha'|\} < \varepsilon < d_0$ ↯

Aim bijection

$$\{\text{roots of } g\} \xleftrightarrow{1:1} \{\text{roots of } f\}$$
$$\beta \longmapsto \alpha(\beta)$$

Claim: $K(\alpha(\beta)) = K(\beta)$

Proof of claim: STP: $\alpha(\beta) \in K(\beta)$

(as $[K(\alpha(\beta)) : K] = n$, $[K(\beta) : K] \leq n$)

$\Rightarrow$ STP: $|\beta - \alpha(\beta)| < |\alpha(\beta) - \alpha'|$

Krasner's la

$\forall \alpha' \neq \alpha(\beta)$ root of $f$

But $|\beta - \alpha(\beta)| < \varepsilon < d_0 \leq |\alpha(\beta) - \alpha'|$

$\forall \alpha' \neq \alpha(\beta)$ root of $f$

Claim: $\beta \mapsto \alpha(\beta)$ is bijection
(after possibly shrinking $\delta$)

Prf of claim: STP: $\beta \mapsto \alpha(\beta)$ injective

Know:

$$|g'(\beta) - f'(\beta)| \leq C_2 \cdot \|f-g\| < C_2 \cdot \delta$$

for some constant $C_2 > 0$ depending only on $f$

Now,

$$|f'(\beta)| \leq \max\{|f'(\beta) - f'(\alpha(\beta))|, |f'(\alpha(\beta))|\}$$

and $|f'(\alpha(\beta))| \geq C_3 > 0$ for some constant $C_3$ (depends on $f$)

If $\delta \to 0$, then $|f'(\beta) - f'(\alpha(\beta))| \to 0$.
as $\beta \to \alpha(\beta)$

$\Rightarrow$ For $\delta$ suff. small

$$|g'(\beta)| = \max\{|g'(\beta)-f'(\beta)|, |f'(\beta)-f'(\alpha(\beta))|, |f'(\alpha(\beta))|\} = |f'(\alpha(\beta))|$$

Upshot: $|g'(\beta)| \geq C_3 > 0$ for all $\delta$ suff. small, all $g$, s.t. $\|f-g\| < \delta$ all roots $\beta$ of $g$

Assume $\alpha_0 := \alpha(\beta) = \alpha(\beta')$ for $\beta \neq \beta'$ roots of $g$

$$\Rightarrow |\beta - \beta'| \leq \max\{|\beta - \alpha_0|, |\beta' - \alpha_0|\} \to 0, \ \delta \to 0$$

& thus

$$|g'(\beta)| = \prod_{\substack{\beta'' \text{ roots of } g \\ \beta'' \neq \beta, \beta'}} \underbrace{|\beta - \beta''|}_{< C_0} \cdot \underbrace{|\beta - \beta'|}_{\to 0, \ \delta \to 0}$$

↯

□

# Finite extensions of complete, discretely valued fields

$K$ complete, disc. valued

$\cup|$

$\mathcal{O}_K \qquad k = \mathcal{O}_K / m_K$

$\cup|$

$m_K = (\pi_K)$

Let

$L$ finite ext. of $K$, $n = [L:K]$

$\cup|$

$\mathcal{O}_L \qquad k_L = \mathcal{O}_L / m_L$

$\cup|$

$m_L = (\pi_L)$

E.g.: $K = \mathbb{Q}_p$, $L = \mathbb{Q}_p(\sqrt[m]{p})$, $\mathbb{Q}_p(\zeta_m)$ $m \geq 1$

Recall: $\mathcal{O}_L$ free over $\mathcal{O}_K$, $\mathrm{rk}_{\mathcal{O}_K} \mathcal{O}_L = n$

Let $v_K : K \to \mathbb{Z} \cup \{\infty\}$ normalized valuation for $K$

$\rightsquigarrow$ unique ext. $v_K : L \to \frac{1}{n} \cdot \mathbb{Z} \cup \{\infty\}$

$x \in L$, $v_K(x) = \frac{1}{n} v_K(N_{L/K}(x))$

$v_L : L \to \mathbb{Z} \cup \{\infty\}$ normalized valuation for $L$

In general: $v_K, v_L : L \to \mathbb{R} \cup \{\infty\}$ are different!

Note: $m_K \cdot O_L = \pi_K \cdot O_L = m_L^e$ for some $e \geq 1$

"ramification index" $e = e(L|K)$

Equiv: $v_K(\pi_L) = \frac{1}{e} \in \frac{1}{n}\mathbb{Z}$ $\quad \left( v_K(\pi_K) = 1 = e \cdot v_K(\pi_L) \right)$

In part, $e \mid n$

Equiv: $v_L(\pi_K) = e$

If $e = 1$ & $k_{L/k}$ is separable, then $L/K$ is called unramified

Def: $f(L|K) := [k_L : k]$ residue degree

Then $n = e(L|K) \cdot f(L|K)$

$\Big( n = \dim_k \big( \mathcal{O}_L / \pi_K \cdot \mathcal{O}_L \big) = e \cdot \dim_k k_L$

as $\mathcal{O}_L / \pi_L \simeq k_L$ and $\mathcal{O}_L / \pi_K$ has filtr.

of length $e$ with quotients $\underbrace{m_L^j / m_L^{j+1}}_{\text{as } \mathcal{O}_L\text{-modules}} \simeq k_L$ $\Big)$

Assume $n = e$ (i.e. "$L/K$ is totally ramified")

$\Rightarrow v_K(\pi_L) = \frac{1}{n} \Rightarrow v_K(N_{L/K}(\pi_L)) = 1$

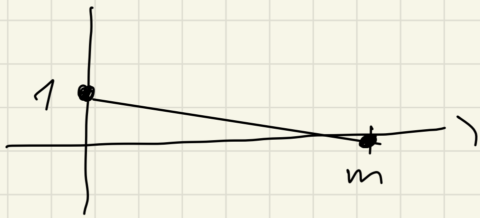
Let $f \in \mathcal{O}_K[x]$ be the min. polynomial of $\pi_L$

$\quad = x^m + a_{m-1} x^{m-1} + \dots + a_0$ $\quad$ ($f \in \mathcal{O}_K[x]$ as $\mathcal{O}_L \cap K = \mathcal{O}_K$)

$\Rightarrow v_K(a_0) = 1$

$\Rightarrow NP(f)$ is a line, namely

fined




, i.e. $v_K(a_i) \geq 1$ for $i = 0, \dots, m-1$

("$f$ is Eisenstein")

As $\mathcal{O}_L = \mathcal{O}_K[\pi_L]$ (check modulo $\pi_K$)

$\Rightarrow$ $m = n$

Converse holds true: $f \in \mathcal{O}_K[x]$ Eisenstein

$\Rightarrow$ $L = K[x]/f(x)$ tot. ramified ext$/K$,

$\mathcal{O}_L = \mathcal{O}_K[\underbrace{\bar{x}}_{\text{res. class of } x \text{ in } L}]$, $\pi_L := \bar{x}$ is a unif.